Three-dimensional solitons and vortices in dipolar Bose-Einstein condensates

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Three-dimensional solitary and vortex structures in Bose-Einstein condensates are studied in the framework of Gross-Pitaevskii model including the simultaneous action of local cubic-quintic nonlinearity and nonlocal dipole-dipole interactions. Nonlocal interactions are shown to change significantly the formation threshold and the numbers of atoms confined into the coherent structures. An appearance of robust high-order ($m = 2$) three-dimensional vortices is revealed.

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I. INTRODUCTION

Recent progress in experimental and theoretical studies of Bose-Einstein condensates (BECs) has opened up a new opportunity to investigate nonlinear interactions of atomic matter waves [1, 2]. By applying an external magnetic field and using the Feshbach resonance, the s-wave scattering length can be tuned, thus it is possible to explore extreme regimes of interaction from strongly repulsive to strongly attractive. The resulting nonlinear evolution of matter waves gives the possibility to observe the nonlinear effects such as atomic self-focusing and formation of solitons.

Dark [3] and bright solitons [4, 5, 6] have already been created in BECs. In the absence of an external trap, such 3D structures are always unstable: they either collapse, if number of atoms exceeds some critical value, or spread out in the opposite case. During an implosion of BEC, the atom density becomes high, and repulsive three-particle interactions come into play and should be taken into account, which gives rise to the additional local quintic nonlinear term in corresponding Gross-Pitaevskii equation (GPE) [9, 10]. In Ref. [11] 3D non-spinning solitons and vortex solitons have been studied in the framework of nonlinear cubic-quintic (CQ) Scrödinger equation with the application to the bulk nonlinear optical media. The competition between the cubic attractive and quintic repulsive terms is able to arrest the collapse and to stabilize 3D solitons. In contrast to non-spinning solitons, vortex solitons can be unstable with respect to azimuthal perturbations which brake 3D vortex into several filaments. In conservative CQ media, single-charged 3D vortices can be stabilized, while higher-order vortices remain unstable [11]. Very recently, stable double-charge vortex solitons were found in the framework of the complex Ginzburg-Landau equation [12]. As the matter of fact, dissipative solitary structures may be more compact and more robust, thus it may be easier to find stable localized vortices in dissipative systems than in their conservative counterparts. In the present paper we present the first example of stable 3D spinning higher-order ($m > 1$) solitons in a conservative nonlinear system. The stabilization is achieved due to the action of the *nonlocal* nonlinearity associated with long-range interparticle interactions in dipolar BEC.

Nonlocal nonlinear media response naturally appears in a wide variety of physical systems such as plasmas [15, 21, 22], Bose- Einstein condensates BEC [26], optical media [25], liquid crystals [23, 24], and soft matter [27]. In the 2D case, nonlocal nonlinear interactions were shown to suppress an azimuthal instability and to stabilize vortex solitons [15, 16, 17, 18]. It was found recently [28] that spinning 3D solitons are unstable against splitting into a set of stable fundamental solitons in the medium with nonlocal thermal nonlinearity.

In degenerate dipolar gases, the nonlocal nonlinearity arises from long-range, partially attractive, and anisotropic dipole-dipole (DD) interactions. The theoretical investigations have shown that the stability of dipolar gases crucially depends on the trap geometry [7, 8]. However, it is not always necessary to include an external trap since self-action of BEC can be enough to provide the conditions for formation of stable localized coherent structures. In the present paper we address the following question: how the nonlocal DD interactions affect the parameters and stability properties of the self-consisted solitons and vortex solitons formed in the trap-free BEC.

II. MODEL EQUATIONS

We consider BEC of atoms with electric dipole d (the model is equally valid for magnetic dipoles) oriented in the z direction by a sufficiently strong external field, and that hence interact via a dipole-dipole potential

$$V_d(\mathbf{r}) = g_d(1 - 3\cos^2\vartheta)/r^3, \tag{1}$$

where $g_d = \alpha d^2/4\pi\varepsilon_0$, ε_0 is the vacuum permittivity, ϑ is the angle formed by the vector joining the interacting particles and the dipole direction. The parameter α can be tuned (see e.g. [26]) in the range $-1/2 \le \alpha \le 1$.

In the mean field approximation, a dipolar BEC at sufficiently low temperatures is described by a GPE with nonlocal nonlinearity:

$$i\hbar\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2M}\Delta\Psi - g\Psi\left|\Psi\right|^2 + g_K\Psi\left|\Psi\right|^4 + g_d\Psi\int V_d(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}',t)|^2 d^3\mathbf{r}' = 0, \tag{2}$$

where $g = 4\pi\hbar^2 a/M$, a is the s-wave scattering length. In the following we consider attractive two-particle interaction ($a < 0$) and repulsive three-particle interaction ($g_K < 0$). Note that we account here only for the conservative part of three-particle interaction. Thus, GPE (2) conserves the norm (the number of atoms in the condensed state):

$$N = \int |\Psi|^2 \, d^3\mathbf{r}, \tag{3}$$

energy:

$$E = \int \left\{ \frac{\hbar^2}{2M}|\nabla\Psi|^2 + \frac{1}{2}g|\Psi|^4 - \frac{1}{3}g_K|\Psi|^6 - \frac{1}{2}g_d|\Psi|^2\Theta \right\} d^3\mathbf{r},\ \Theta = \int V_d(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2 d^3\mathbf{r}', \tag{4}$$

momentum and angular momentum.

In the next section, the general properties of stationary solitons and vortices are studied by analytical variational method and numerically.

III. STATIONARY 3D SOLITONS AND VORTEX SOLITONS

Stationary solutions of the Eq. (2) have the form $\Psi(\mathbf{r},t) = \psi(\mathbf{r})\exp(i\lambda t)$ and obey the dimensionless equation:

$$-\lambda\psi + \Delta\psi + \psi\,|\psi|^2 - \psi\,|\psi|^4 + C\psi\,\Theta = 0, \tag{5}$$

$$\Theta = \hat{F}^{-1}\left[\tilde{V}_d(k)\hat{F}\left\{|\psi|^2\right\}\right], \tag{6}$$

where $\hat{F}$ denotes the Fourier transformation, $\tilde{V}_d(k) = \frac{4\pi}{3}(3k_z^2/k^2 - 1)$ is the Fourier transform of dipole-dipole interaction potential. Applying the following rescaling transformations: $\mathbf{r}_{\text{dimless}} = (2Mg^2/\hbar^2 g_K)^{1/2}\mathbf{r}_{\text{phys}}$, $\psi_{\text{dimless}} = |g_K/g|^{1/2}\psi_{\text{phys}}$, the number of free parameters is reduced to two, namely $C = g_d/g$, and $\lambda = -\mu|g_K|/g^2$ (μ being the chemical potential).

In order to study the general properties of stationary 3D solitary and vortex solutions analytically, we employed the variational analysis with the following ansatz:

$$\Psi(\mathbf{r},t) = h\left(\frac{\rho}{a_\rho}\right)^{|m|}\exp\left\{-\frac{\rho^2}{2a_\rho^2} - \frac{z^2}{2a_z^2} + im\varphi\right\}, \tag{7}$$

where $\rho = \sqrt{x^2+y^2}$, φ is azimuthal angle. Integer m is the topological charge, $m = 0$ corresponds to non-spinning solitons, $m > 0$ – to vortex solitons. Two variational parameters $a_\rho(t)$ and $a_z(t)$ describe radii of the structure in the directions across and along the external field, and the amplitude $h(t) = N^{1/2}\pi^{-3/4}a_\rho^{-1}(t)\left[a_z(t)m!\right]^{-1/2}$ is found from normalization condition (3). Using the trial function (7) one can obtain the energy functional (4) at fixed number of atoms N as the function of two variational parameters a_ρ and a_z. These calculations were performed analytically, but the resulting expressions are too cumbersome to be presented here. Stationary solutions correspond to the stationary points of the energy functional E, i.e. the parameters a_ρ and a_z are found from the set of equations: $\partial E/\partial a_z$=0, $\partial E/\partial a_\rho = 0$ at fixed N. The dependencies $N(\lambda)$ obtained by variational method below are compared with the numerical results.

To find the stationary solutions of Eq. (5) numerically, the stabilized iteration procedure similar to that proposed by Petviashvili (see, e.g. [29]) was implemented on a fully 3D grid with the resolutions up to 128^3. Figure 1 shows the isosurfaces of constant $|\psi|^2$ for vortex solitons $m = 2$ at different C. The change in sign of nonlocality parameter C leads to flattening ($C > 0$) or to elongation ($C < 0$) of the distribution of atoms along z-direction compared to the case $C = 0$. Below we shall concentrate mainly at the case $C > 0$.

In the Fig. 2 the numbers of atoms versus the parameter λ for solitons ($m = 0$) and vortices ($m = 1, 2$) are shown. The numerically found states are plotted by circles, while the dashed lines present analytical dependencies calculated by the variational method. It is seen that the variational predictions are in a good agreement with the results of numerical simulations. The divergence between analytical and numerical curves increases as the number of atoms N confined into the structure goes up, and this can be explained by the observation, that with increase of λ, soliton and vortex solutions develop more and more pronounced plateau and their profiles then strongly deviate from the trial Gaussian profile used in the variational approach. In the Fig. 3, numbers of atoms are plotted versus the parameter λ for different strengths of non-locality for the vortices $m = 1$ (dependencies $N(\lambda)$ for solitons $m = 0$ and vortices $m = 2$ show very much similar behavior). One can see that increased nonlocality leads to the significant decrease of the structure's formation threshold and to the substantial elongation of the range of accessible parameter λ.

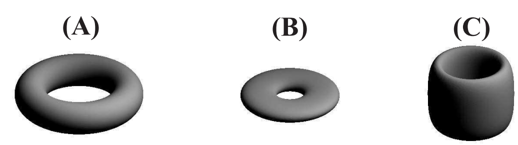

FIG. 1: Numerically found stationary vortex solutions (isosurfaces of constant $|\psi|^2$) for $m = 2$ and different values of the nonlocality parameter C: (A) $C = 0$, $\lambda = 0.1$, (B) $C = 0.3$, $\lambda = 0.45$, (C) $C = -0.3$, $\lambda = 0.08$.

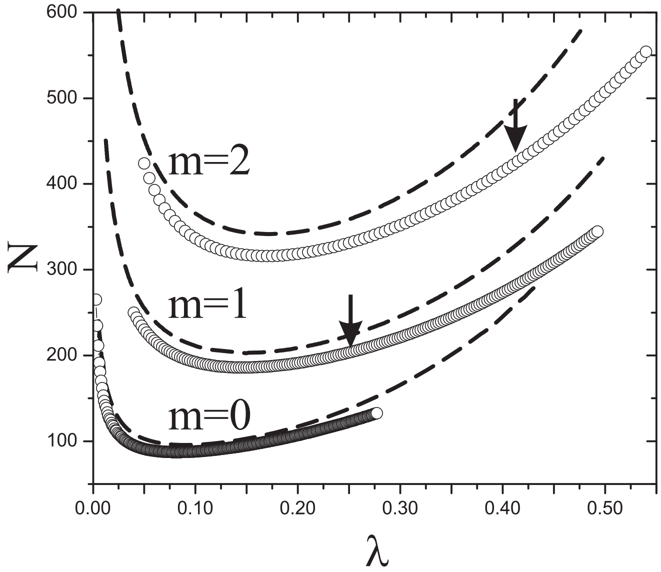

FIG. 2: Number of particles N vs λ for different stationary solutions at $C = 0.3$. Numerical results are shown in circles, dashed lines are for variational predictions. The arrows indicate stability thresholds for vortices.

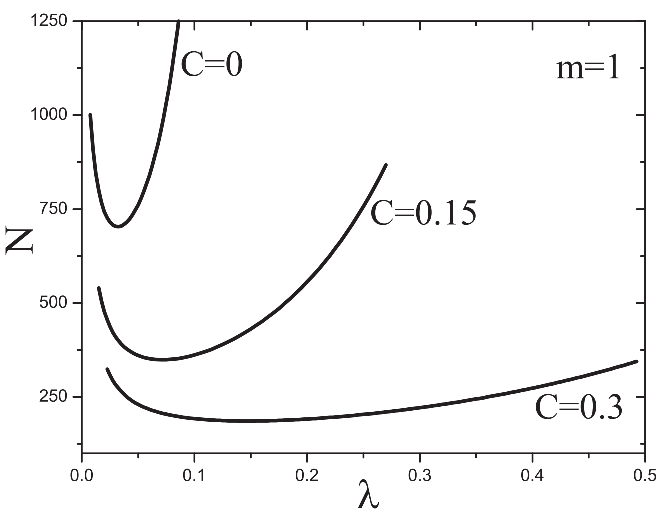

FIG. 3: Number of particles N vs parameter λ for vortices with $m = 1$ with different C. Numerical results.

IV. STABILITY AND DYNAMICS

We start investigation of the stability of vortices with analysis of small perturbations applied to the stationary solution:

$$\Psi = (\psi + \epsilon)e^{i\lambda t},$$

where $|\epsilon(\mathbf{r}, t)| << |\psi|$. Linearizing the GPE (2) in vicinity of stationary solution one gets the nonstationary equation which describes evolution of small perturbation:

$$i\frac{\partial \epsilon}{\partial t} - \lambda\epsilon + \Delta\epsilon + (2|\psi|^2\epsilon + \psi^2\epsilon^*)$$
$$-(2\psi^2\epsilon^* + 3|\psi|^2\epsilon)|\psi|^2 + C(\delta\Theta\,\psi + \Theta\epsilon) = 0, \qquad (8)$$

where Θ is given by Eq. (6) and

$$\delta\Theta = \hat{F}^{-1}\left[\tilde{V}_d(k)\hat{F}\left\{\psi\epsilon^* + \psi^*\epsilon\right\}\right].$$

We have solved Eq. (8) numerically by means of split-step technic. The azimuthal instability growth rate was calculated as follows:

$$\gamma = \frac{1}{2\Delta t}\ln\left\{\frac{\nu(t+\Delta t)}{\nu(t)}\right\},$$

where Δt is the time step, and $\nu(t) = \langle\epsilon|\epsilon\rangle$ is the norm of the perturbation. When the perturbation grows exponentially, $\gamma(t)$ saturates at some value γ, which was taken as the maximum growth rate.

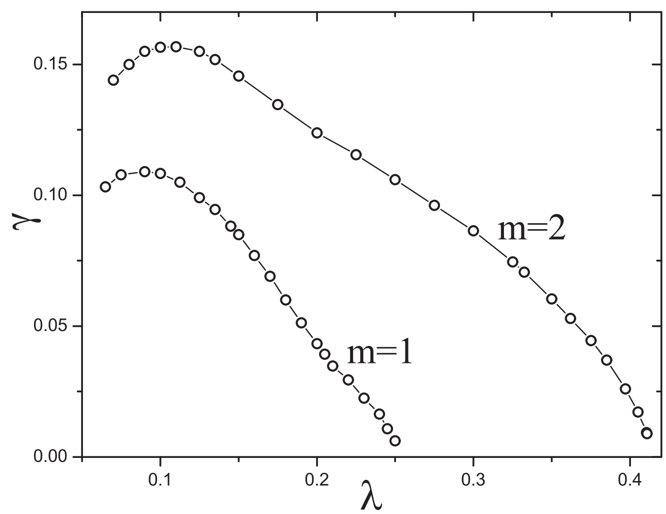

FIG. 4: Maximum growth rates γ vs λ for vortices with $m = 1$ and $m = 2$, C=0.3

The soliton structures having zero topological charge are azimuthally stable, and their stability region coincides with that described by the Vakhitov-Kolokolov criterion ($\partial N/\partial\lambda > 0$) [19]. The maximum growth rates obtained by means of above approach are given in Fig. 4 for vortices $m = 1$ and $m = 2$ at $C = 0.3$.

Note that our analysis is formulated in very general form, it gives the *maximum* growth rate at given λ but not the growth rate of specific unstable azimuthal mode. This explains the knees in the dependencies $\gamma(\lambda)$ (see Fig. 4) which correspond to the intersections of growth rates $\gamma_L(\lambda)$ for different azimuthal numbers L of perturbations.

As it is known (see, e.g., [30]), the competition between cubic and quintic nonlinear terms leads to the formation of solitary structures having a kind of plateau on the top when number of atoms N is high enough. Further increase of N leads to saturation of the amplitude and to

elongation of the plateau. In the previous works [13, 14], stabilization of the 2D vortices was associated with the appearance of such plateau. It is remarkable that in the BECs described by the GP model with additional nonlocal nonlinearity, stabilization of vortices occurs *before* the plateau is formed. At given nonlocality parameter C, the growth rates vanish at some threshold value of λ due to the action of nonlocal dipole-dipole interactions. For instance, at $C = 0.3$, the single-charged vortices get the stability window at $\lambda > 0.26$ and double-charged vortices ($m = 2$) – at $\lambda > 0.42$, these threshold values are marked by the arrows in Fig. 2. As for vortices with $m > 2$, no stabilization was observed.

The evolution and the dynamical stability of solitary and vortex structures was simulated numerically using the well-known split-step Fourier method [20], where the nonlocal DD integral term was calculated in the spectral space. We used perturbed solutions found by the stationary solver as an initial condition for the nonstationary problem. For the structures with plateau, Petviashvili's approach may fail and is unable to reproduce the $N(\lambda)$ diagram completely in the high-energy region. In this case, we used the appropriate initial condition with the parameters extracted from the variational analysis as an input for the non-stationary solver to check an existence and stability of solitons and vortices for values of N where our relaxative solver is inefficient.

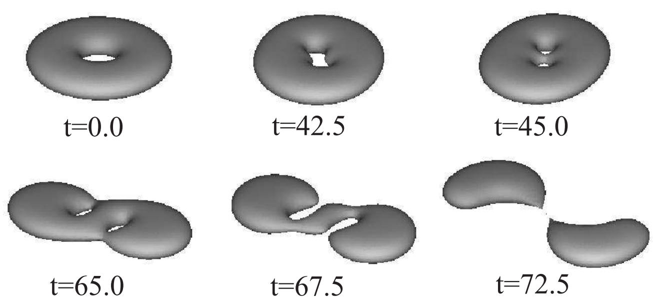

FIG. 5: Decay of unstable vortex ($m = 2$, $C = 0.3$, $\lambda = 0.25$): isosurfaces of constant particle density $|\psi|^2$ are given for different times.

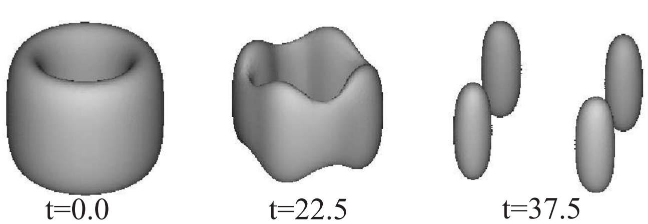

FIG. 6: Decay of unstable vortex ($m = 2$, $C = -0.3$, $\lambda = 0.08$): isosurfaces of constant particle density $|\psi|^2$ are given for different times.

Direct numerical simulations confirmed the predictions obtained by the linear stability analysis. Solitons were found to be stable everywhere in the region $\partial N/\partial\lambda > 0$, stability thresholds for vortices coincide with those predicted by the linear stability analysis. In the unstable region, destruction of a vortex may occur in the different ways, depending on the number of atoms confined into the structure and on the applied perturbation. Typical unstable evolution snapshots are plotted in Fig. 5 and Fig. 6 for different signs of nonlocality parameter C.

V. CONCLUSIONS

In the present paper we study the influence of nonlocal dipole-dipole interparticle interaction on 3D solitons and vortices in BEC described by GPE with local contact attractive two-particle and repulsive three-particle interaction. Nonlocal anisotropic dipole-dipole interaction leads to quantitative, as well as to qualitative changes in scales, energies and stability properties of supported solitary structures. The formation threshold (minimum number of atoms, which is needed to form a structure) goes down compared to the case of absence of DD interactions, and possible range of chemical potentials significantly elongates. While soliton structures having zero topological charge are stable if $\partial N/\partial\lambda > 0$, vortices may exhibit azimuthal instability. We proved the stabilization of 3D vortices with $m = 1$ and $m = 2$ and revealed their stabilization thresholds. All higher order ($m > 2$) vortices were found to be unstable. Up to our knowledge, this is a first example of 3D stable vortex with the topological charge larger than one in a conservative system.

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